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The compressible inviscid leading-edge vortex

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The conically symmetric solution of the Eulerian equations of an incompressible fluid obtained by Hall, thought to be descriptive of flow properties in a leading-edge vortex, is generalized to include the effects of compressibility.

1. Introduction

An interesting feature of the flow over a slender delta wing at incidence is the presence of leading-edge vortices which are formed by the rolling-up of the shear layer that separates from a leading edge. A theoretical study of the core structure has been made by Hall (1961), who found approximations to the velocity and pressure distributions by dividing the flow into a convective outer part and a diffusive subcore. For the outer region viscous effects were ignored, and, as the flow throughout was assumed to be axisymmetric, this resulted in conical velocity and pressure fields. For the inner part of the core approximations analogous to those of boundary-layer theory were made on the assumption that changes in the radial direction occur much more rapidly than those in the axial direction for a slender core. The join of the two solutions was satisfactory from a practical point of view, and a paper by Stewartson & Hall (1963) has since made the match mathematically valid by defining appropriate variables to describe the flow in the inner layer. The flow throughout was taken to be incompressible, continuous and rotational.

The effect of viscosity on the incompressible leading edge vortex having thus been examined, it therefore seemed of interest to examine the effect of compressibility on the inviscid vortex, and this topic is the subject of this paper. The simplified model of the core is similar to that adopted by Hall in that it is geometrically slender, the velocity and pressure fields are steady and axially symmetric, and the flow is continuous and therefore rotational as it includes no vortex sheet. When the viscous terms of the Navier–Stokes equations are ignored, a solution in which all the dependent variables are functions of a conical parameter θ alone is possible everywhere. The governing equations have an energy integral, and the flow must be homentropic. The assumption of slenderness ($\theta^2 \ll 1$) implies that the system of equations may be reduced to

$$\rho^{\gamma-1}\left(1-\frac{\gamma-1}{2}\beta\frac{d\rho}{du}\right)=\frac{\gamma-1}{2}M_2^2(A^2-u^2), \quad -\theta\frac{du}{d\theta}=\beta\rho, \tag{1.1}$$

where ρ, u are the reduced density and axial component of velocity, γ is the ratio of the specific heats, and other undefined quantities are constants. It is shown that, for given axial and azimuthal velocities and Mach number at $\theta = \theta_2$, the outer edge of the core, an acceptable solution is possible for only one value of the radial velocity at $\theta = \theta_2$. This solution extends to the axis $\theta = 0$ if $1 < \gamma \leqslant 2$, but, if $\gamma > 2$, the density vanishes at a non-zero value of θ.

Equations (1.1) are solved numerically for a range of values of the constants M_2 and A when $\gamma = 1{\cdot}4$, and the results are presented graphically. The variations of the pressure and velocity components with angular co-ordinate indicate that for small Mach number compressibility effects are confined to the immediate neighbourhood of the axis. In the final section an analytical solution for the problem when the density variation is small is determined by dividing the vortex into an almost incompressible outer region and a relatively slender subcore in which compressibility effects are greater. For the outer core a solution is sought which has the inviscid solution found by Hall as a first approximation, and an expansion is obtained in powers of ϵ which is defined to be equal to the square of the Mach number based on the axial velocity at the outer edge of the core. As expected, this solution is not valid in the immediate neighbourhood of the axis of the core where compressibility effects are greater, and it is necessary to introduce an inner solution analogous to the boundary layer of the viscous problem. There is no difficulty in effecting the matching procedure for all values of the constant γ, and for $1 < \gamma \leqslant 2$ this compressible layer, which is of thickness $O(\epsilon^{\frac{1}{2}})$, has the desired effect of extending the solution to the axis, on which the velocity components take finite values and the density is zero. However, if $\gamma > 2$, this inner solution exhibits a singular behaviour before the axis is reached which is in agreement with the conclusions obtained by direct consideration of equations (1.1).

It is concluded that for $\gamma \leqslant 2$ a non-singular solution exists throughout the vortex, and that for small values of ϵ there is a boundary-layer effect of compressibility analogous to that of viscosity as discussed by Hall.

2. The equations of motion

When the velocity and pressure fields are axially symmetric, the equations describing the flow of a compressible inviscid fluid are, in cylindrical co-ordinates (r, x),

continuity:
$$\frac{\partial(\rho' u')}{\partial x} + \frac{\partial(\rho' w')}{\partial r} + \frac{\rho' w'}{r} = 0; \tag{2.1}$$

momentum:
$$u' \frac{\partial u'}{\partial x} + w' \frac{\partial u'}{\partial r} = -\frac{1}{\rho'} \frac{\partial p'}{\partial x}, \tag{2.2}$$

$$u' \frac{\partial v'}{\partial x} + w' \frac{\partial v'}{\partial r} + \frac{v' w'}{r} = 0, \tag{2.3}$$

$$u' \frac{\partial w'}{\partial x} + w' \frac{\partial w'}{\partial r} - \frac{v'^2}{r} = -\frac{1}{\rho'} \frac{\partial p'}{\partial r}; \tag{2.4}$$

entropy:
$$u' \frac{\partial S'}{\partial x} + w' \frac{\partial S'}{\partial r} = 0; \tag{2.5}$$

and the equation of state:
$$p' = \rho' \mathscr{R} T'. \tag{2.6}$$

Here u' and w' are the axial and radial velocity components, v' is the circumferential velocity component, and p', ρ', S', T' are the pressure, density, entropy and temperature.

Equations (2.1)–(2.6) admit of a solution in which all the dependent variables are functions of the conical parameter $\theta = r/x$ alone, and in terms of this as independent variable (2.1)–(2.5) become

$$-\theta \frac{d}{d\theta}\left\{\rho'\left(u'-\frac{w'}{\theta}\right)\right\}+2\frac{\rho' w'}{\theta} = 0, \tag{2.7}$$

$$-\left(u'-\frac{w'}{\theta}\right)\frac{du'}{d\theta} = \frac{1}{\rho'}\frac{dp'}{d\theta}, \tag{2.8}$$

$$-\theta\left(u'-\frac{w'}{\theta}\right)\frac{dv'}{d\theta}+\frac{v'w'}{\theta} = 0, \tag{2.9}$$

$$-\theta\left(u'-\frac{w'}{\theta}\right)\frac{dw'}{d\theta}-\frac{v'^2}{\theta} = -\frac{1}{\rho'}\frac{dp'}{d\theta}, \tag{2.10}$$

$$(u'-w'/\theta)\, dS'/d\theta = 0. \tag{2.11}$$

The boundary conditions for the problem are that the values of u', v', ρ' are prescribed at the outer edge of the vortex, denoted by $\theta = \theta_2$, and that w' is zero on the axis of the core, so there are no sources or sinks. Thus, in the notation employed by Hall,

$$\theta = 0, \quad w' = 0; \quad \theta = \theta_2, \quad u' = U_2, \quad v' = V_2, \quad \rho' = R_2. \tag{2.12}$$

Two integrals of equations (2.7)–(2.11) follow immediately. From (2.7) and (2.9) we obtain

$$\frac{v'^2}{V_2^2} = \frac{\rho'(u'-w'/\theta)}{R_2(U_2-W_2/\theta_2)}, \tag{2.13}$$

where W_2 is the value of the radial velocity w' at the outside edge of the vortex. The expression (2.13) is written as

$$v'^2 = \beta'\rho'(u'-w'/\theta), \quad \text{where} \quad \beta' = V_2^2/R_2(U_2-W_2/\theta_2). \tag{2.14}$$

It should be noted that the value of W_2 and hence of β' cannot be determined until the boundary condition on w' on the axis $\theta = 0$ has been imposed on the solution, though we may infer that it is positive for an acceptable solution since $(1+\theta_2^2)^{-\frac{1}{2}}(\theta_2 U_2 - W_2)$ is the component of velocity perpendicular to the generators at the outer edge of the core, and we intuitively expect a flow in which the vortex is fed by fluid from outside. Equation (2.11) implies that the flow is homentropic so that the pressure is a function of the density alone, and the equation of state (2.6) may be written in the form

$$p' = k'\rho'^{\gamma}, \tag{2.15}$$

where k' is the constant of proportionality.

Equations (2.7)–(2.10) also possess an energy integral. From (2.8), (2.10) and (2.14) we first obtain

$$\frac{du'}{d\theta}+\theta\frac{dw'}{d\theta}+\frac{\beta'\rho'}{\theta} = 0, \tag{2.16}$$

and then, from (2.7), (2.8), (2.16),

$$\frac{1}{2}\frac{d}{d\theta}(u'^2+v'^2+w'^2)+\frac{1}{\rho'}\frac{dp'}{d\theta} = 0, \tag{2.17}$$

which, using (2.15), may be integrated as

$$\tfrac{1}{2}(u'^2+v'^2+w'^2)+\frac{\gamma}{\gamma-1}\frac{p'}{\rho'} = H, \tag{2.18}$$

where H is constant.

An interesting result, which is a property of any such homenergic homentropic flow, is, as noted by Howarth (1956), that the vortex lines are parallel to the streamlines. For the problem under consideration it is easily deducible from the equations of motion. If we denote the vector (u', v', w') by $\mathbf{q}$ and write down the components of the vector product $\mathbf{q}\wedge\operatorname{curl}\mathbf{q}$, the condition for each to vanish holds by virtue of equations (2.8)–(2.10) and (2.14). Hall noted that the vortex lines were approximately parallel to the spiralling streamlines in his discussion of the incompressible vortex to terms $O(\theta^2)$, the physical significance of this and the more general result obtained here being that there exists a circumferential component of vorticity which induces a high velocity along the axis.

The governing equations are now rewritten on the assumption that terms $O(\theta^2)$ may be neglected. This slenderness condition was first employed by Hall (1961) and implies that the term $\theta(dw'/d\theta)$ may be omitted from (2.16). It is then possible to write all the dependent variables as functions of $\log\theta$. Defining the non-dimensional variables

$$\phi = \log\theta_2/\theta, \quad u = u'/U_2, \quad \rho = \rho'/R_2, \quad w(\phi) = w'/(U_2\theta), \tag{2.19}$$

equations (2.7) and (2.16) become

$$d\{\rho(u-w)\}/d\phi+2\rho w = 0, \tag{2.20}$$

$$du/d\phi = \beta\rho, \tag{2.21}$$

where $\beta = \beta' R_2/U_2$. The third equation to complete the system is that obtained by neglecting the term w'^2 in the energy equation (2.18); thus

$$\tfrac{1}{2}u^2+\tfrac{1}{2}\beta\rho(u-w)+\{k\gamma/(\gamma-1)\}\rho^{\gamma-1} = \tfrac{1}{2}A^2. \tag{2.22}$$

Here $k = k'(R_2^{\gamma-1}/U_2^2)$, and the constant A^2, proportional to H, is given by

$$A^2 = 1+\frac{V_2^2}{U_2^2}+\frac{2k\gamma}{\gamma-1}. \tag{2.23}$$

The physical significance of the parameter k is that it is inversely proportional to the square of the Mach number based on the axial velocity at the outside edge of the core since it follows from (2.15) that $k = (\gamma M_2^2)^{-1}$.

The order of the above system of equations may again be reduced by one. The radial velocity w is eliminated between (2.20) and (2.22) to give

$$\rho^{\gamma-1}\left[1-\frac{\gamma-1}{2}\beta\frac{d\rho}{du}\right] = \frac{\gamma-1}{2}M_2^2(A^2-u^2), \tag{2.24}$$

equation (2.21) enabling u to be written as the independent variable. The boundary conditions associated with equations (2.20), (2.21) and (2.24) are

$$u = 1, \quad \rho = 1 \quad \text{when} \quad \phi = 0, \tag{2.25}$$

together with the condition on w' at the axis which implies that $e^{-\phi}w \to 0$ as $\phi \to \infty$.

3. The properties of the governing equations

The purpose of this section is to discuss the properties of equations (2.21), (2.24), and we shall show that, if the axial and azimuthal velocities and the density, are prescribed at the outer edge of the core, then a physically acceptable solution exists for only one value of the radial velocity at $\theta = \theta_2$. This statement implies a unique value of the constant β. It will also be demonstrated that, if $1 < \gamma \leqslant 2$, this solution extends to the axis, but, if $\gamma > 2$, the density vanishes at a non-zero value of θ, and the vortex has a vacuum core. Such values of γ are not however realized in practice.

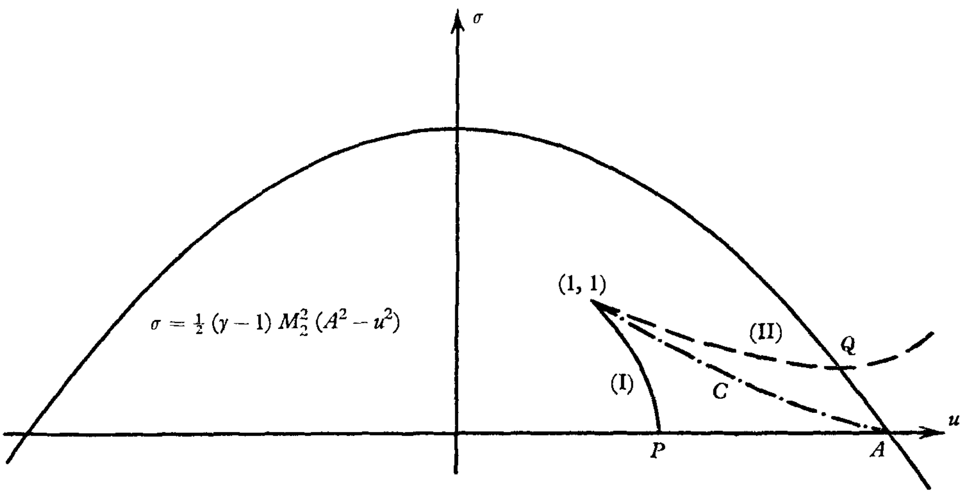

FIGURE 1. The integral curves of equation (3.1).

For further consideration of equation (2.24) it is convenient to write $\sigma = \rho^{\gamma-1}$, and then we obtain

$$\beta \frac{d\sigma}{du} = \frac{2\sigma - (\gamma-1)M_2^2(A^2-u^2)}{\sigma^{1/(\gamma-1)}}, \tag{3.1}$$

where $\sigma^{1/(\gamma-1)}$ is defined only for $\sigma \geqslant 0$. The fundamental curves which are aids to sketching the characteristics of equation (3.1) are $\sigma = 0$, on which $d\sigma/du$ is infinite, and the parabola $\sigma = \frac{1}{2}(\gamma-1)M_2^2(A^2-u^2)$, on which $d\sigma/du$ is zero. These curves are shown in figure 1, which also illustrates the two possible types of integral curves which can occur in the half-plane $\sigma \geqslant 0$ for different values of the positive constant β. Each integral curve starts from the point (1, 1), which lies inside the parabola, and it follows from (2.21) that the required direction is in the sense of u increasing. When $\sigma < \frac{1}{2}(\gamma-1)M_2^2(A^2-u^2)$, $d\sigma/du < 0$, and the integral curves are either of type (I) or type (II) as shown. It follows from consideration of (3.1) that no two solutions corresponding to distinct values of β intersect again after the point (1, 1). Thus between these two families of curves must lie a dividing curve C going through the point $u = A$, and we now demonstrate that this, with its associated value of β, is the only acceptable solution. This is achieved by considering the curves (I), (II), C successively.

(a) *Curves of type I*

These curves are excluded by consideration of the character of the point P at which $\rho = 0$ but $u \neq A$. Suppose that, at P, $u = u_0$ and $\rho \sim (u_0 - u)^q$ where $0 < q < 1$ since $d\rho/du$ is infinite. Equation (2.24) gives the value of q as $1/\gamma$, and

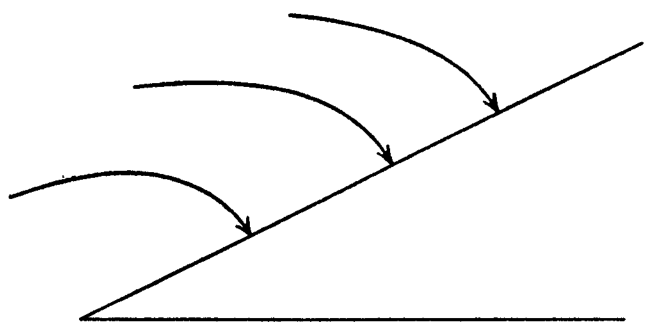

FIGURE 2. Streamlines corresponding to solutions of type (I).

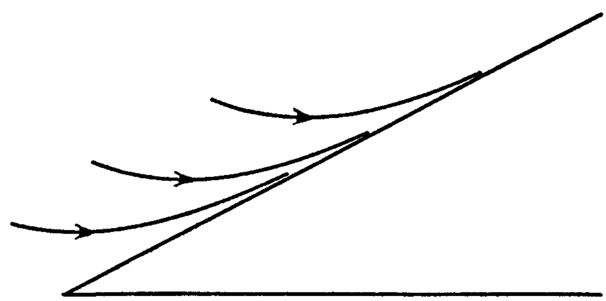

FIGURE 3. Streamlines corresponding to solutions of type (II).

such a solution is excluded on noting that the velocity perpendicular to the generators, given by (2.20)–(2.22) as

$$u - w = -k\gamma\rho^{\gamma-2}(d\rho/du), \tag{3.2}$$

has $(u_0 - u)^{-1/\gamma}$ as its asymptotic behaviour as $u \to u_0$. From (2.21) it follows that the value of ϕ at which this occurs is finite, and this radial line is therefore a 'limit line' which the streamlines reach with infinite velocity. The situation, which is physically unacceptable, is illustrated in figure 2.

(b) *Curves of type II*

At the point Q of figure 1, $w = u$ since $d\rho/du = 0$. The streamlines, which in the neighbourhood of Q have an equation of the form $x \propto (\phi_Q - \phi)^{-\frac{1}{2}}$, where ϕ_Q denotes the value of ϕ at Q, are shown in figure 3 and become tangential to the line $\phi = \phi_Q$, which they meet only at infinity. The fluid in the cone bounded by this stream surface may be at rest, but, as it has not come from any main stream external to the vortex, it is difficult to envisage how such a flow could be set up. Also experimental evidence indicates that the leading-edge vortex consists of a sink-type flow unlike the case under consideration, which we thus infer cannot occur.

(*c*) *The curve C*

It remains to show that the solution given by the curve C is appropriate to the problem. Near $u = A$ let us assume that $\rho \sim (A-u)^r$ where $r > 0$.Then, since

$$\frac{d\rho}{du} = \frac{1}{\gamma-1}\sigma^{(2-\gamma)/(\gamma-1)}\frac{d\sigma}{du},$$

and at A $d\sigma/du$ is finite and non-zero, we see that $r \gtreqless 1$ according as $\gamma \lesseqgtr 2$. Equation (2.24) shows that near $u = A$

$$(A-u)^{r(\gamma-1)}[1+\tfrac{1}{2}(\gamma-1)\beta r(A-u)^{r-1}] \sim (\gamma-1)M_2^2(A-u)\,2A, \qquad (3.3)$$

from which we deduce that, if $\gamma < 2$, $r = 1/(\gamma-1)$, if $\gamma = 2$, $r = 1$; while, if $\gamma > 2$, $r = 2/\gamma$, all of which are consistent with the above assertion. The value ϕ_A of ϕ corresponding to the point $u = A$ is now examined. From (2.21) we obtain

$$du/d\phi \sim \beta(A-u)^r, \qquad (3.4)$$

of which the integral is

$$(A-u)^{1-r} \sim \beta(r-1)\phi + \text{const.}, \qquad (3.5)$$

except if $r = 1$, in which case

$$\log(A-u) \sim -\beta\phi + \text{const.} \qquad (3.6)$$

The results (3.5), (3.6) show that, if $\gamma \leqslant 2$, the axial velocity u does not attain the value A until $\phi \to \infty$, and the axis is reached; but, if $\gamma > 2$, ϕ_A is finite, and the density becomes zero at a non-zero value of the angular co-ordinate. It follows from (3.2) that, for the latter range of γ, $w = u = A$ when $\phi = \phi_A$, which is the condition for the cone $\phi = \phi_A$ to be a stream surface, and the vortex has a vacuum core. However, if $1 < \gamma \leqslant 2$, there is no such singularity, and the curve C represents the required solution of equation (2.24). In the following section the numerical solution of this equation is considered when $\gamma = 1{\cdot}4$, the accepted value for air.

In § 5, where an analytical solution for small Mach number is given, we find support for the ideas introduced here.

4. The numerical solution for $\gamma = 1{\cdot}4$

We now investigate solutions of (2.24) numerically with the condition $\rho = 1$, $u = 1$. The parameters available are the Mach number M_2 and the swirl ratio V_2/U_2, both of which may be prescribed arbitrarily to describe the conditions at the outer edge of the core. The value of β must then be chosen so that, as demonstrated in the previous section, the solution obtained is the one which passes through the point $u = A$, $\rho = 0$. Due to the difficulty of predicting this value with any accuracy, and to the instability of the equation as the required point is neared, the semi-inverse method described below is adopted.

If in equation (2.24) we write

$$u = AX, \quad (\gamma-1)\beta\rho/2A = Y, \tag{4.1}$$

we have

$$1 - dY/dX = s(1-X^2)/Y^{\gamma-1}, \tag{4.2}$$

where

$$s = \tfrac{1}{2}(\gamma-1)M_2^2A^2[2A/(\gamma-1)\beta]^{1-\gamma}. \tag{4.3}$$

The required solution now terminates at $X = 1$, $Y = 0$ (corresponding to the axis), and we take this as the starting point for the numerical integration. The procedure adopted was as follows. With $s = s_0$, say, (4.2) was integrated to $X = a\,(0 < a < 1)$, $Y = b$; the appropriate values of M_2, V_2/U_2 and β corresponding to this solution were derived from

$$\text{(i)}\ \ s_0 = (\gamma-1)M_2^2b^{\gamma-1}/2a^2, \quad \text{(ii)}\ \ A = a^{-1}, \quad \text{(iii)}\ \ 2A/(\gamma-1)\beta = b^{-1}, \tag{4.4}$$

where the latter two are derived from conditions at the outer edge of the core. These equations may be solved for M_2, V_2/U_2 and β remembering that

$$A^2 = 1 + \frac{V_2^2}{U_2^2} + \frac{2}{(\gamma-1)M_2^2}.$$

To obtain the variation of the axial velocity u with angular co-ordinate, the equation to be solved is (2.21) in the form

$$d\phi/dX = (\gamma-1)/2Y, \tag{4.5}$$

with $\phi = 0$ when $X = a$, the outer edge of the core.

Equation (4.2) has been integrated numerically for various values of $s \leqslant 1$ when $\gamma = 1{\cdot}4$, using the fact that, near $X = 1$, Y is of the form

$$Y^{0\cdot 4} = s(1-X^2)\{1 - 5s^{\frac{5}{2}}X(1-X^2)^{\frac{3}{2}} + 125s^5X^2(1-X^2)^3 - \tfrac{25}{2}s^5(1-X^2)^4 - (5^5\,.\,13/8)\,s^{\frac{15}{2}}X^3(1-X^2)^{\frac{9}{2}} + \ldots\}. \tag{4.6}$$

In figures 4, 5, 6 are plotted axial and circumferential velocity profiles and pressure distributions derived from the cases $s = \frac{1}{2}, 1$, where the abscissa θ/θ_2 represents angular displacement from the axis. The function employed to illustrate the pressure, namely $(p'-P_2)/R_2V_2^2$, where P_2 denotes the value of p' at the outer edge of the core, is not the most convenient for demonstrating the fact that p' vanishes on the axis, but was chosen to facilitate direct comparison with figure 7 of Hall's (1961) paper for zero Mach number when the pressure tends to minus infinity as the axis is reached; the corresponding profile for zero Mach number is included in each diagram. Although every $s = s_0$ yields an infinite number of solutions for arbitrary V_2/U_2, in each profile this parameter has been taken as unity since Hall has demonstrated that the effect of varying swirl ratio is of degree rather than character. It should be noted that the gradient of the pressure, like that of the other functions considered, is infinite on the axis, although this is not readily apparent due to the scale chosen.

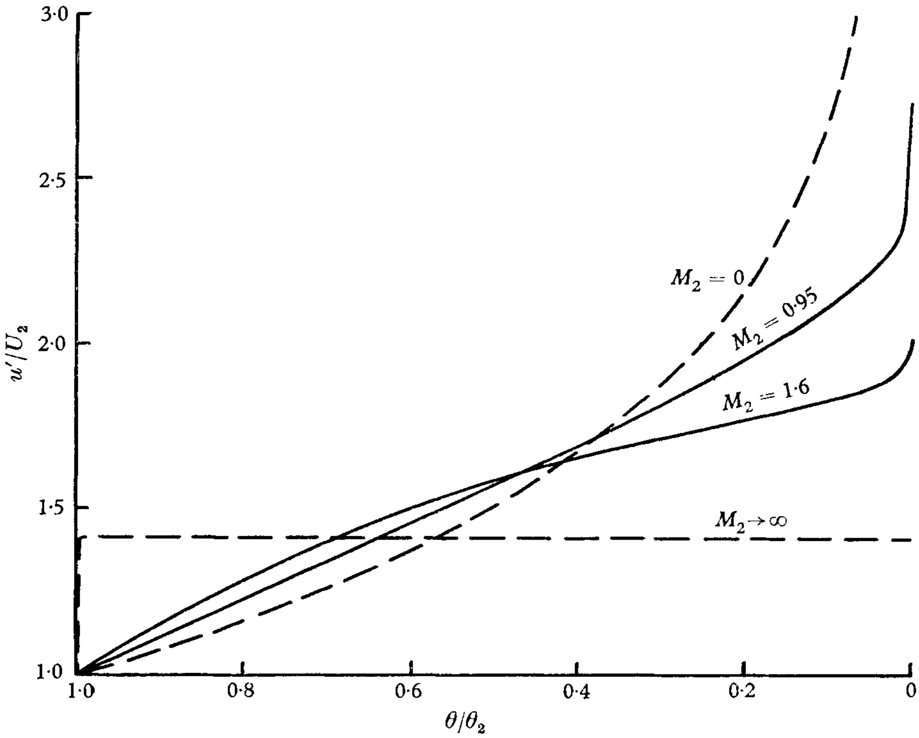

FIGURE 4. Axial velocity profiles showing the effect of varying M_2.

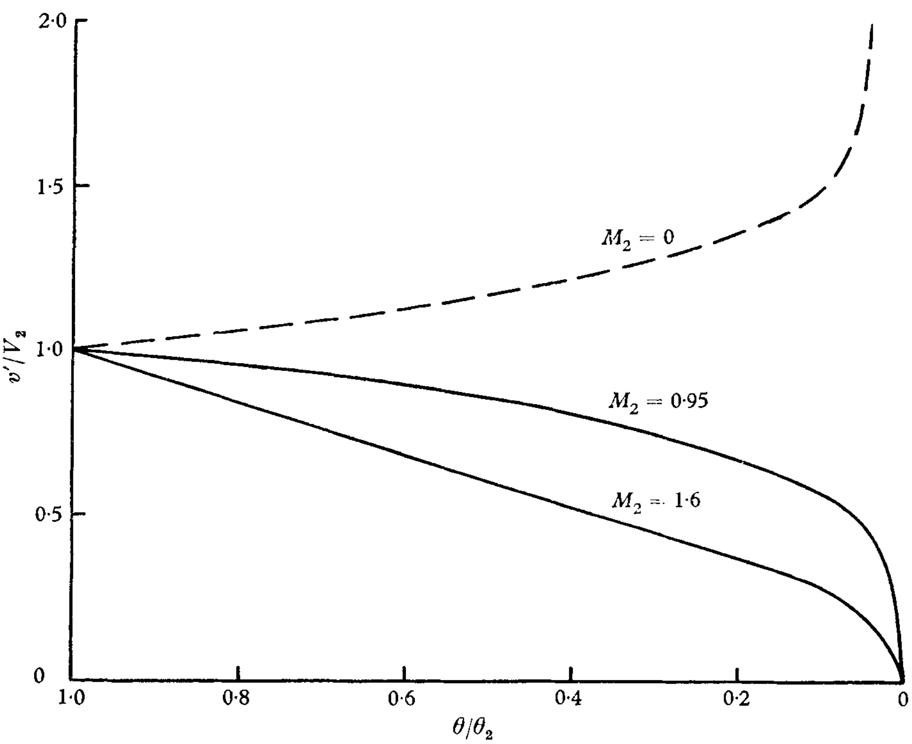

FIGURE 5. Circumferential velocity profiles showing the effect of varying M_2.

With $s = \frac{1}{2}$ and $V_2/U_2 = 1$, the relations (4.4) lead to $A = 2{\cdot}74$, the value of u on the axis, $M_2 = 0{\cdot}95$, $\beta = 1{\cdot}12$. The other quantity of interest is the value of the radial velocity at the outer edge of the core which is obtained as $W_2/U_2\theta_2 = 0{\cdot}10$. With $s = 1$ the corresponding values are $A = 1{\cdot}98$, $M_2 = 1{\cdot}6$, $\beta = 1{\cdot}69$ and $W_2/U_2\theta_2 = 0{\cdot}41$. The quantities associated with the curve for zero Mach number are $s = 0$, $\beta = 0{\cdot}732$, $W_2/U_2\theta_2 = -0{\cdot}366$, while $A \to \infty$. The sign of the radial velocity in the different cases is interesting. When the Mach number is zero, it is, as shown by Hall, negative throughout the core; in the two cases considered here with the Mach number $O(1)$ it was found to be positive throughout though with $w < u$ so that each streamline does in fact cross every generator it meets, while

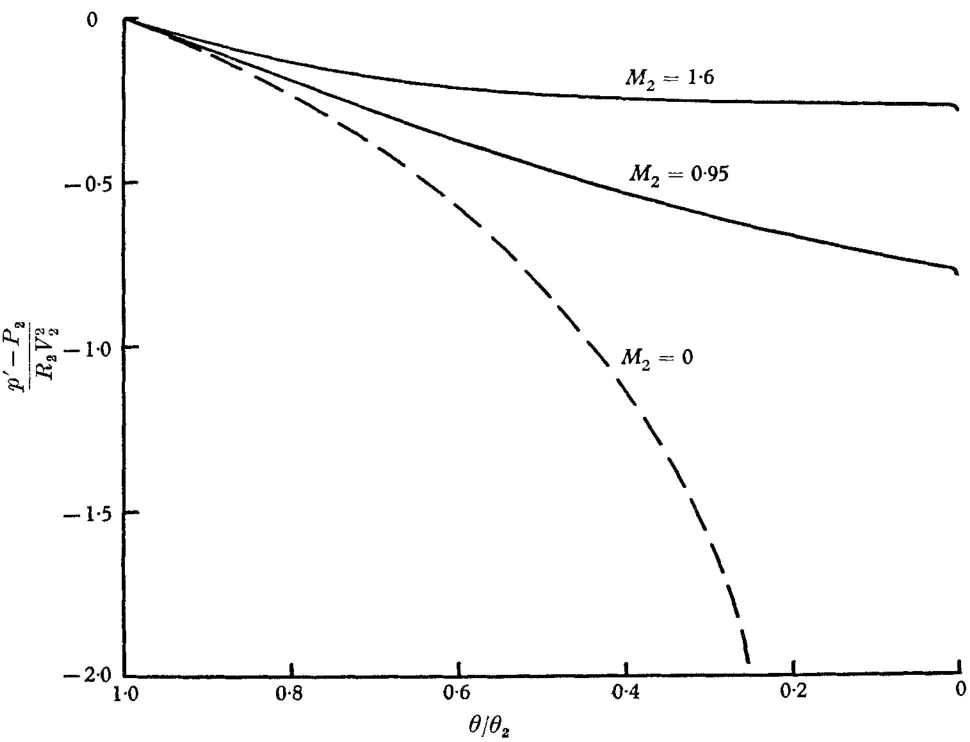

FIGURE 6. Pressure distributions in the vortex showing the effect of varying M_2.

in the final section it is shown that for small Mach number w changes sign from negative to positive in the neighbourhood of the axis. Perhaps the most unexpected feature of the velocity profiles is that the circumferential velocity in figure 5 sinks to zero on the axis in contrast to the incompressible case.

Both figures 4 and 5 illustrate that for small Mach number compressibility has a 'boundary-layer' effect on the flow in the immediate neighbourhood of the axis, analagous to the diffusive subcore discussed by Hall. The similarity between the compressible and viscous vortices may be emphasized here. In both cases the axial velocity attains a finite value on the axis of the core while the circumferential velocity is reduced to zero. One difference is that for the compressible vortex the pressure, like the density, is zero on the axis, but is non-zero for the viscous vortex. However, the essential features are the same. In the next section the 'boundary-layer' phenomenon noted above enables analytical solutions of the governing equations to be obtained as power series in M_2, for $M_2 \ll 1$, by means of a matching procedure between an outer 'incompressible' layer and an inner 'compressible' layer.

For large M_2 it follows from equation (2.24) that

$$\beta\rho^{\gamma} = \tfrac{1}{3}\gamma M_2^2(A-u)^2(2A+u), \tag{4.7}$$

and hence that

$$\beta = \tfrac{1}{3}\gamma M_2^2(A-1)^2(2A+1). \tag{4.8}$$

Thus as $M_2 \to \infty$, $\beta \to \infty$, and also $s \to \infty$. In addition $W_2 \to U_2\theta_2$, and the streamlines only just enter the vortex. It can be shown by consideration of equation (2.21) that the velocity components and density attain their axial values in the immediate neighbourhood of the outer edge of the vortex so we now have a 'boundary-layer' phenomenon exhibited at $\theta = \theta_2$. The profile as $M_2 \to \infty$ is included in figure 4 and gives a qualitative assessment of the accuracy of the profiles for finite Mach number.

5. The solution for small M_2

When M_2 is small, the effects of compressibility are, as noted in the previous section, confined to a narrow region in the neighbourhood of the axis, enabling the solution to be determined analytically throughout the vortex by standard 'boundary-layer' techniques. The most convenient equations to consider are (2.20), (2.21) and (2.8), which, when written in terms of the variables defined by (2.19), becomes

$$M_2^2(u-w)\,du/d\phi + \rho^{\gamma-2}\,d\rho/d\phi = 0. \tag{5.1}$$

If we let $M_2 \to 0$ in this equation, it follows that in the limit the density ρ is constant and equal to its value at the outside edge of the core. The solutions for the velocity components are then as found by Hall (1961), and, as this is to be regarded as a first approximation to the complete compressible solution, the appropriate parameter for a series expansion is $\epsilon = M_2^2$. To find a solution of (2.20), (2.21) and (5.1) in powers of ϵ, we write

$$\rho = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots, \tag{5.2}$$

with similar expansions for u, w, β, and in terms of these functions the boundary conditions are, from (2.25),

$$u_0 = 1, \quad \rho_0 = 1, \quad u_i = 0, \quad \rho_i = 0 \quad (i \geqslant 1), \quad \text{when} \quad \phi = 0, \tag{5.3}$$

$$e^{-\phi}w_i \to 0 \quad \text{as} \quad \phi \to \infty \quad \text{for all } i. \tag{5.4}$$

The expansion for β is necessary as this constant also depends on the compressibility.

When the series of the form (5.2) are substituted into (2.20), (2.21), (5.1), the the terms of lowest order in ϵ give

$$d\{\rho_0(u_0-w_0)\}/d\phi + 2\rho_0 w_0 = 0, \tag{5.5}$$

$$du_0/d\phi = \beta_0\rho_0, \tag{5.6}$$

$$d\rho_0/d\phi = 0, \tag{5.7}$$

and the solution of these satisfying the boundary conditions (5.3), (5.4) is

$$\rho_0 = 1, \quad u_0 = 1+\beta_0\phi, \quad w_0 = -\tfrac{1}{2}\beta_0. \tag{5.8}$$

Using (2.14), the value of β_0 is obtained as

$$\beta_0 = (1+2V_2^2/U_2^2)^{\frac{1}{2}}-1 > 0, \tag{5.9}$$

and (5.8), (5.9) are identical with the solution given by Hall for the outer inviscid region of the vortex.

The equations satisfied by ρ_1, u_1, w_1 follow from (2.20), (2.21), (5.1) on equating to zero the coefficients of ϵ. From (5.1) we obtain, using (5.8),

$$\rho_0^{\gamma-2}(d\rho_1/d\phi) = -\beta_0(1+\tfrac{1}{2}\beta_0+\beta_0\phi); \tag{5.10}$$

thus, integrating, we have

$$\rho_1 = -\beta_0[(1+\tfrac{1}{2}\beta_0)\,\phi+\tfrac{1}{2}\beta_0\phi^2]. \tag{5.11}$$

Then (2.21) gives, making use of (5.11),

$$du_1/d\phi = -\beta_0^2[(1+\tfrac{1}{2}\beta_0)\,\phi+\tfrac{1}{2}\beta_0\phi^2]+\beta_1, \tag{5.12}$$

and, since $u_1(0) = 0$,

$$u_1 = -\tfrac{1}{2}\beta_0^2[(1+\tfrac{1}{2}\beta_0)\,\phi^2+\tfrac{1}{3}\beta_0\phi^3]+\beta_1\phi. \tag{5.13}$$

The equation for w_1 is obtained from (2.20) as

$$-dw_1/d\phi+2w_1 = \beta_0[(1+\tfrac{1}{2}\beta_0)^2+3\beta_0(1+\tfrac{1}{2}\beta_0)\,\phi+\tfrac{3}{2}\beta_0^2\phi^2]-\beta_1, \tag{5.14}$$

using (5.8), (5.10), (5.13). The solution of (5.14) satisfying the condition that $e^{-\phi}w_1 \to 0$ as $\phi \to \infty$ is

$$w_1 = \tfrac{1}{2}\beta_0(1+\tfrac{5}{2}\beta_0+\tfrac{7}{4}\beta_0^2)-\tfrac{1}{2}\beta_1+\tfrac{3}{2}\beta_0^2(1+\beta_0)\phi+\tfrac{3}{4}\beta_0^3\phi^2. \tag{5.15}$$

Equating coefficients of ϵ in (2.14) gives the value of β_1 in terms of β_0 as

$$\beta_1(1+\beta_0) = \tfrac{1}{2}\beta_0^2(1+\tfrac{5}{2}\beta_0+\tfrac{7}{4}\beta_0^2), \tag{5.16}$$

and so (5.15) may be written

$$w_1 = (\beta_1/\beta_0)\,(1+\tfrac{1}{2}\beta_0)+\tfrac{3}{2}\beta_0^2(1+\beta_0)\,\phi+\tfrac{3}{4}\beta_0^3\phi^2. \tag{5.17}$$

The equations and solutions for ρ_2, u_2, w_2 follow in a similar manner. As the details are tedious, we merely quote the value of β_2 which is given by

$$\begin{aligned}\beta_2(1+\beta_0) = {} & \tfrac{1}{4}\beta_0^3[\gamma(1+\tfrac{9}{2}\beta_0+\tfrac{15}{2}\beta_0^2+\tfrac{37}{8}\beta_0^3)-\tfrac{3}{2}\beta_0(5+\tfrac{31}{2}\beta_0+13\beta_0^2)] \\ & -\tfrac{3}{2}\beta_0\beta_1(1+\tfrac{5}{6}\beta_0-\tfrac{5}{12}\beta_0^2)+\beta_1^2(1+\tfrac{1}{2}\beta_0)/\beta_0. \end{aligned} \tag{5.18}$$

It is at once apparent that, however small the value of ϵ, the solution to equations (2.20), (2.21), (5.1) given by (5.2) with the above values of ρ_0, u_0, w_0, ρ_1, u_1, w_1 will not be valid in the immediate neighbourhood of the axis since the polynomial solutions tend to infinity with ϕ. The reason for this is that compressibility effects are greater in this region and cannot be neglected even to the first order as is implied by (5.5)–(5.7). The appropriate variables for describing the flow in this inner zone are now determined, and the solution in the inner zone must be matched with the present solution in their common region of validity

The independent variable for this region will be denoted by ψ, and it is obtained by multiplying ϕ by an appropriate power of ϵ so that in the limit, as $\epsilon \to 0$, a matching procedure may be effected between the outer and inner solutions as $\phi \to \infty$ and $\psi \to 0$. The scaling factors required to multiply ϕ and the dependent variables ρ, u, w are determined by consideration of the forms of ρ, u, w obtained from the outer solution in the double limit $\epsilon \to 0$, $\phi \to \infty$ and from the requirement that at least one of the terms in M_2^2 in (5.1) shall be of the same order as $\rho^{\gamma-2}d\rho/d\phi$ in the inner layer as $\epsilon \to 0$. We write

$$\psi = \epsilon^m \phi, \quad \bar{\rho} = \epsilon^q \rho, \quad \bar{u} = \epsilon^r u, \quad \bar{w} = \epsilon^s w, \tag{5.19}$$

where m, q, r, s are constants, and $\bar{\rho}, \bar{u}, \bar{w}$ are independent of ϵ to the first order. In the limit of zero ϵ, (5.8) shows that for large ϕ, $\rho \sim 1$, $u \sim \beta_0 \phi$, $w \sim -\frac{1}{2}\beta_0$, and we have at once that $q = s = 0$ and $r = m$. Making the transformation (5.19) in equation (5.1) gives

$$\epsilon(\epsilon^{-m}\bar{u} - \bar{w})\, d\bar{u}/d\psi + \epsilon^m \bar{\rho}^{\gamma-2} d\bar{\rho}/d\psi = 0, \tag{5.20}$$

showing that the only acceptable value of m is $\frac{1}{2}$; this has the desired effect of giving the same order in ϵ to $\bar{u}\, d\bar{u}/d\psi$ and $\bar{\rho}^{\gamma-2} d\bar{\rho}/d\psi$, the remaining term being of higher order. With this value of m, when written in terms of the new variables given by (5.19), equations (2.20), (5.1), (2.21) become

$$d\{\bar{\rho}(\bar{u} - \epsilon^{\frac{1}{2}}\bar{w})\}/d\psi + 2\bar{\rho}\bar{w} = 0, \tag{5.21}$$

$$(\bar{u} - \epsilon^{\frac{1}{2}}\bar{w})\, d\bar{u}/d\psi + \bar{\rho}^{\gamma-2} d\bar{\rho}/d\psi = 0, \tag{5.22}$$

$$d\bar{u}/d\psi = \beta\bar{\rho}. \tag{5.23}$$

Equations (5.21)–(5.23) are the appropriate form of the equations of motion for the description of the flow in the inner core of the vortex. As $\epsilon \to 0$, they become

$$d(\bar{\rho}_0 \bar{u}_0)/d\psi + 2\bar{\rho}_0 \bar{w}_0 = 0, \tag{5.24}$$

$$\bar{u}_0\, d\bar{u}_0/d\psi + \bar{\rho}_0^{\gamma-2} d\bar{\rho}_0/d\psi = 0, \tag{5.25}$$

$$d\bar{u}_0/d\psi = \beta_0 \bar{\rho}_0, \tag{5.26}$$

where the expansion of β in powers of ϵ is given by (5.2), and the coefficients β_i are determined by the outer solution. The boundary conditions associated with equations (5.24)–(5.26) are that, as $\psi \to 0$, $\bar{\rho}_0$, $\bar{u}_0$, $\bar{w}_0$ must match with ρ_0, $\epsilon^{\frac{1}{2}}u_0$, w_0 as $\phi \to \infty$. Equation (5.25) may be integrated at once to give

$$\tfrac{1}{2}\bar{u}_0^2 + \bar{\rho}_0^{\gamma-1}/(\gamma-1) = \text{const.} = \tfrac{1}{2}\Gamma^2 \quad \text{say}, \tag{5.27}$$

and, since for large ϕ we have from (5.8) that $\rho_0 \sim 1$, $u_0 \sim \beta_0 \phi$, it follows that for small ψ, $\bar{\rho}_0 \sim 1$, $\bar{u}_0 \sim \beta_0 \psi$. This gives the value of Γ^2 as

$$\Gamma^2 = 2/(\gamma-1). \tag{5.28}$$

Substituting for $\bar{\rho}_0$ from (5.26) into (5.27) we obtain

$$\bar{u}_0^2 + \frac{2}{\beta_0^{\gamma-1}(\gamma-1)}\left(\frac{d\bar{u}_0}{d\psi}\right)^{\gamma-1} = \Gamma^2, \tag{5.29}$$

so that $\quad d\bar{u}_0/d\psi = D(\Gamma^2 - \bar{u}_0^2)^{1/(\gamma-1)}, \quad$ where $\quad D = \beta_0(\tfrac{1}{2}\gamma - \tfrac{1}{2})^{1/(\gamma-1)}, \qquad (5.30)$

and, integrating,

$$D\psi = \int_0^{\bar{u}_0} \frac{dt}{(\Gamma^2 - t^2)^{1/(\gamma-1)}}, \tag{5.31}$$

since $\bar{u}_0(0) = 0$.

It is at this stage that it first becomes evident that the flow pattern in the vortex core depends critically on the value of γ. It follows from its definition as the ratio of the specific heats that $\gamma > 1$, and from (5.27) we see that for a meaningful value of the first approximation to the density $\bar{u}_0 \leqslant \Gamma$. Then (5.31) gives the maximum value of ψ as

$$D\psi_s = \int_0^\Gamma \frac{dt}{(\Gamma^2 - t^2)^{1/(\gamma-1)}}, \tag{5.32}$$

which is infinite if $1/(\gamma-1) \geqslant 1$ but is finite if $1/(\gamma-1) < 1$, and the integral is convergent. Thus $\bar{\rho}_0$ remains non-zero until the axis is reached if $1 < \gamma \leqslant 2$ but becomes zero at a finite value of ψ if $\gamma > 2$. This result is not unexpected since in § 4 it was shown that the flow extended to the axis if $\gamma \leqslant 2$, but, if $\gamma > 2$, the density became zero at a non-zero value of the angular co-ordinate. The value Γ of $\bar{u}_0$ attained on the axis when $\gamma \leqslant 2$ indicates that the correct integral curve was found in § 3 where it was asserted that the axial value of u was equal to A since for small M_2 we have $A^2 = 2/\{(\gamma-1)M_2^2\}$.

The integral in (5.32) can be evaluated in terms of known functions for certain values of γ, and in particular, if $\gamma = 1{\cdot}4$, it may be written

$$D\psi = \frac{1}{\Gamma^4}\left[\frac{\bar{u}_0}{(\Gamma^2-\bar{u}_0^2)^{\frac{1}{2}}} + \frac{1}{3}\frac{\bar{u}_0^3}{(\Gamma^2-\bar{u}_0^2)^{\frac{3}{2}}}\right], \tag{5.33}$$

and the fact that $\psi \to \infty$ as $\bar{u}_0 \to \Gamma$ is at once apparent.

The value of $\bar{w}_0$ may now be obtained from (5.24), and it is found that

$$\bar{w}_0 = -\tfrac{1}{2}\beta_0(\bar{\rho}_0 - \bar{u}_0^2/\bar{\rho}_0^{\gamma-2}), \tag{5.34}$$

and this automatically satisfies the condition that $\bar{w}_0 \to -\frac{1}{2}\beta_0$ as $\psi \to 0$ at the outer edge of the inner layer. If $1 < \gamma < 2$, then $\bar{w}_0$ is zero on the axis; if $\gamma = 2$, its value there is β_0; though, if $\gamma > 2$, and $\bar{\rho}_0$ vanishes for $\psi = \psi_s$, then $\bar{w}_0 \to \infty$ as $\psi \to \psi_s$, which is again in agreement with § 4 where it was demonstrated that at the end point of the solution $w = u$, and $u = O(\epsilon^{-\frac{1}{2}})$. Another interesting feature of the solution (5.34) is that $\bar{w}_0$ changes sign from negative to positive when $\gamma\bar{\rho}_0^{\gamma-1} = \bar{u}_0^2$. This means that in the neighbourhood of the axis the radial flow is outwards, though small, but is towards the axis in the remainder of the vortex.

To take account of the terms in $\epsilon^{\frac{1}{2}}$ in equations (5.21)–(5.23) we expand the dependent variables as power series in $\epsilon^{\frac{1}{2}}$ and continue the matching procedure outlined above between the solution for the outer part of the vortex and the inner core, in which compressibility effects are important. We write

$$\bar{\rho} = \bar{\rho}_0 + \epsilon^{\frac{1}{2}}\bar{\rho}_1 + \epsilon\bar{\rho}_2 + \dots \tag{5.35}$$

with similar expansions for $\bar{u}$ and $\bar{w}$, where $\bar{\rho}_0$, $\bar{u}_0$, $\bar{w}_0$ are given by (5.27), (5.31), (5.33). Substituting (5.35) into (5.21)–(5.23), the terms in $\epsilon^{\frac{1}{2}}$ give

$$\frac{d}{d\psi}\{\bar{\rho}_0(\bar{u}_1 - \bar{w}_0)\} + \frac{d}{d\psi}(\bar{\rho}_1\bar{u}_0) + 2(\bar{\rho}_0\bar{w}_1 + \bar{\rho}_1\bar{w}_0) = 0, \tag{5.36}$$

$$\bar{u}_0\frac{d\bar{u}_1}{d\psi} + (\bar{u}_1 - \bar{w}_0)\frac{d\bar{u}_0}{d\psi} + \frac{d}{d\psi}(\bar{\rho}_0^{\gamma-2}\bar{\rho}_1) = 0, \tag{5.37}$$

$$d\bar{u}_1/d\psi = \beta_0\bar{\rho}_1. \tag{5.38}$$

Using (5.26) and (5.24) successively, (5.37) may be written in the form

$$\frac{d}{d\psi}(\bar{u}_0\bar{u}_1)+\frac{d}{d\psi}(\bar{\rho}_0^{\gamma-2}\bar{\rho}_1) = -\tfrac{1}{2}\beta_0\frac{d}{d\psi}(\bar{\rho}_0\bar{u}_0), \tag{5.39}$$

which, when integrated, becomes

$$\bar{u}_0\bar{u}_1+\bar{\rho}_0^{\gamma-2}\bar{\rho}_1 = -\tfrac{1}{2}\beta_0\bar{\rho}_0\bar{u}_0+A_1/\beta_0, \tag{5.40}$$

where A_1 is a constant of integration to be determined by matching with the appropriate terms of the outer solution. Substituting for $\bar{\rho}_1$ in (5.40) from (5.38) we obtain a first order differential equation for $\bar{u}_1$. It is

$$\frac{d\bar{u}_1}{d\psi}+\frac{\beta_0\bar{u}_0}{\bar{\rho}_0^{\gamma-2}}\bar{u}_1 = -\tfrac{1}{2}\beta_0^2\frac{\bar{u}_0}{\bar{\rho}_0^{\gamma-3}}+\frac{A_1}{\bar{\rho}_0^{\gamma-2}}. \tag{5.41}$$

Making use of (5.25), (5.26), an integrating factor of this equation is found to be $(\bar{\rho}_0)^{-1}$, so performing the integration we have

$$\frac{\bar{u}_1}{\bar{\rho}_0} = \tfrac{1}{2}\beta_0\log\bar{\rho}_0+A_1\int_0^{\psi}\frac{d\psi}{\bar{\rho}_0^{\gamma-1}}+B_1, \tag{5.42}$$

where B_1 is a second constant of integration. Writing the outer solution for u in terms of the variable ψ shows that, as $\psi\to 0$, we require $\bar{u}_0 \sim \beta_0\psi$, which is satisfied by (5.31), and

$$\bar{u}_1 \sim 1-\tfrac{1}{2}\beta_0^2(1+\tfrac{1}{2}\beta_0)\psi^2, \tag{5.43}$$

$$\bar{u}_2 \sim \beta_1\psi. \tag{5.44}$$

Considering the expansion of $\bar{\rho}_0$ for small ψ obtained from (4.27), (4.31), and imposing the boundary conditions $\bar{u}_1 = 1$, $d\bar{u}_1/d\psi = 0$ at $\psi = 0$, gives $A_1 = 0$ and $B_1 = 1$. Thus we obtain from (5.42)

$$\bar{u}_1 = \bar{\rho}_0(\tfrac{1}{2}\beta_0\log\bar{\rho}_0+1), \tag{5.45}$$

and from (5.40)

$$\bar{\rho}_1 = -\bar{u}_0\bar{\rho}_0^{2-\gamma}(\bar{u}_1+\tfrac{1}{2}\beta_0\bar{\rho}_0). \tag{5.46}$$

The following expression for $\bar{w}_1$ is easily obtained from (5.36):

$$\bar{w}_1 = \frac{1}{2}\frac{\beta_0\bar{u}_0}{\bar{\rho}_0^{\gamma-2}}\{\bar{u}_1[3+(\gamma-2)\bar{u}_0^2/\bar{\rho}_0^{\gamma-1}]+\beta_0\bar{\rho}_0[3+(\gamma-3)\bar{u}_0^2/\bar{\rho}_0^{\gamma-1}]\}. \tag{5.47}$$

Examining the behaviour of $\bar{\rho}_1, \bar{u}_1, \bar{w}_1$ in the neighbourhood of the axis as $\psi\to\infty$ we see that in the non-singular case $1 < \gamma \leqslant 2$, since $\bar{\rho}_0\to 0$, $\bar{u}_0\to\Gamma$ as $\psi\to\infty$, all three functions tend to zero except when $\gamma = 2$, in which case $\bar{w}_1\to -\beta_0^2\sqrt{2}$. However, when $\gamma > 2$, $\bar{\rho}_0 = 0$ when $\psi = \psi_s$, and, as ψ nears this value, $\bar{w}_1$ tends to infinity.

The third term in the expansion for the inner solution has also been found though the details are omitted. The values attained by the functions on the axis are that as $\psi\to\infty$, $\bar{\rho}_2\to 0$, $\bar{u}_2\to(1/4\Gamma)(\beta_0^2+2\beta_0+2)$, $\bar{w}_2\to 0$, which are again in agreement with the expansions for small Mach number of the axial values of these functions deduced in §3. If $\gamma > 2$, a similar singular behaviour to that indicated above is exhibited by all three functions as $\psi\to\psi_s$.

The conclusion is then that, if γ lies in the range $1 < \gamma \leqslant 2$, which includes $\gamma = 1{\cdot}4$, a perfectly regular solution can be found throughout the core which is divided into an 'incompressible' outer layer together with a 'compressible' inner layer of thickness $O(\epsilon^{\frac{1}{2}})$. The density remains non-zero until the axis is reached, and in the neighbourhood of the axis the axial velocity u is $O(\epsilon^{-\frac{1}{2}})$ and so is finite for non-zero ϵ. The convergence of the expansions in ϵ obtained is slow, but formally there should be no difficulty in obtaining as many terms of the series as required.

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